$$X_{pi} = \int_0^l \rho x \phi_{pi}(x)\,dx; \quad p = 1,2,3; \quad i = 1,\ldots,N_p \tag{11f}$$

$$G_{122imn} = \int_0^l EA\,\phi'_{1i}(x)\phi'_{2m}(x)\phi'_{2n}(x)\,dx$$

$$i = 1,\ldots,N_1; \quad m,n = 1,\ldots,N_2 \tag{11g}$$

$$G_{133imn} = \int_0^l EA\,\phi'_{1i}(x)\phi'_{3m}(x)\phi'_{3n}(x)\,dx$$

$$i = 1,\ldots,N_1; \quad m,n = 1,\ldots,N_3 \tag{11h}$$

$$E_{ppqqijkl} = \int_0^l EA\,\phi'_{pi}(x)\phi'_{pj}(x)\phi'_{qk}(x)\phi'_{ql}(x)\,dx$$

$$p,q = 2,3; \quad i,j = 1,\ldots,N_p; \quad k,l = 1,\ldots,N_q \tag{11i}$$

Results and Discussion

The spin-up problem in Ref. 1 is also considered here to study the effect of nonlinear structural terms. The beam is initially at rest and an angular velocity is given along the a_3 axis at the base. Only the in-plane motions are excited in line with the assumptions. The beam parameters are as follows: $E = 68{,}950{,}000{,}000\ \text{N/m}^2$, $\rho = 1.2\ \text{kg/m}^3$, $A = 0.0004601\ \text{m}^2$, $I_3 = 0.0000002031\ \text{m}^4$, and $l = 10$ m. The angular velocity history is taken to be identical with that in Ref. 1

$$w_3 = 6/15\left[t - \frac{15}{2\pi}\sin\frac{2\pi t}{15}\right]\ \text{rad/s}, \qquad 0 \le t \le 15\ \text{s}$$

$$w_3 = 6\ \text{rad/s}, \qquad t \ge 15\ \text{s} \tag{12}$$

The transverse mode shapes are taken as the fixed-free nonrotating eigenfunctions of a uniform beam under transverse vibration, whereas the longitudinal modes are taken as the eigenfunctions of a fixed-free uniform rod under longitudinal vibrations. The axial and transverse motions are represented by one and three modes, respectively. The axial and bending responses of the tip of the cantilever beam resulting from the formulation presented in this Note are shown in Figs. 2a and 2b, respectively. The solid curves correspond to the analysis of this Note, where all higher-order terms have been retained, and the dashed curves refer to the situation where all second- and third-order terms are eliminated. In the nonlinear analysis, the transverse deflection grows initially in a direction opposite to that of the base motion. After reaching a maximum displacement, the tip goes back toward the equilibrium position and settles down to a steady oscillation. The nonlinear stiffening action in the beam prevents it from going unstable, and the very absence of such terms causes the linear beam to diverge away from the equilibrium point. A foreshortening effect is observed in the axial response (Fig. 2a) when the beam is started from rest. The final axial response in the case of nonlinear analysis is a steady oscillation about a nonzero equilibrium point that corresponds to the steady-state axial displacement under the centrifugal force field. The effect of centrifugal stiffening has been studied by numerous investigators in the past, and a brief discussion on the subject can be found in Ref. 4. Likins et al.[5] have assumed steady-state axial displacements and moderate rotations to obtain the stiffening effect. Vigneron[6] has assumed foreshortening of the beam and uses Hamilton's principle to show that the centrifugal stiffening terms arise from the kinetic energy terms. In Ref. (4), Kaza and Kvaternik have observed that foreshortening of the beam need not be considered explicitly if terms up to fourth order are retained in the energy terms. The present formulation has taken such an approach and does not assume any a priori kinematical restriction on the displacement field. The axial and transverse displacement fields have been chosen to be independent of each other, and the foreshortening of the beam is a consequence of the imposed base motion.

Conclusions

In this Note, we have formulated the problem of a cantilever beam attached to a moving support by using Kane's method. The formulation is valid for large displacements, and all geometric nonlinearities have been considered in the strain-displacement relations. The method has been validated by studying the stability characteristics of a beam under the spin-up maneuver. It has been demonstrated that structural nonlinearities play a major role in the transient response characteristics and they cannot be ignored.

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Gravitational Moment Exerted on a Small Body by an Oblate Body

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Introduction

THE gravitational forces and moments that act on an orbiting body are well recognized as important influences on the motion of such a body. This paper illustrates a method for finding an analytic expression for the moment about a body's mass center produced by gravitational forces.

Expressions for the gravitational moment exerted on a body by a sphere (or particle) appear in numerous places, including Ref. 1. The equation in Ref. 1 is particularly simple because it does not express the gravitational moment in terms of a particular unit vector basis. This simplicity is made possible by expressing the gravitational moment in terms of a unit vector and a dyadic. One can implement the equation by expressing the unit vector and dyadic in any convenient basis.

In his Engineering Note, Glandorf[2] seeks and develops a method for obtaining the gravitational moment exerted by bodies other than spheres. Kane, et al.[1] suggest an alternative to the method proposed by Glandorf and simplified by Wilcox.[3] Use of the method from Ref. 1 can lead to vector-dyadic expressions that are simple in appearance and basis-independent.

As an example of the use of the method suggested in Ref. 1, this paper derives an expression for the gravitational moment

Received Dec. 21, 1987.

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Assertion

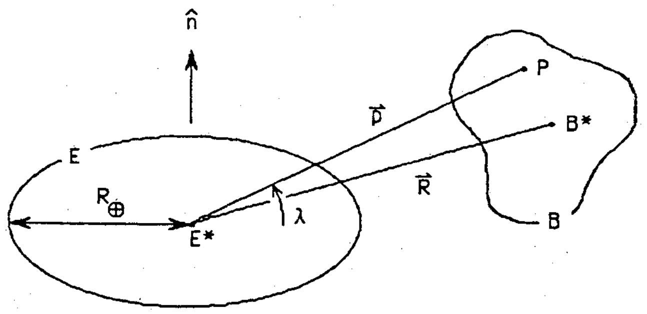

Fig. 1 Small body in the presence of an oblate spheroid.

exerted by an oblate spheroid. Roberson[4] has the objective of developing an equivalent expression. However, Roberson's work quickly becomes complicated because the derivation is tied to a particular basis from the outset.

Also included is an example in which the equation for gravitational moment exerted by an oblate spheroid on a body the size of a space station is numerically evaluated.

Assertion

Figure 1 shows a small body B in the presence of body E. The distance between the mass centers B^* and E^* of B and E is assumed to exceed the greatest distance from B^* to any point of B. Body E is assumed henceforward to be an oblate spheroid. The system of gravitational forces exerted by E on B produces a moment about B^* that is given approximately by

$$M = \frac{3\mu_\oplus}{R^3}\hat{r}\times\underline{I}\cdot\hat{r} + \frac{\mu_\oplus J_2 R_\oplus^2}{2R^5}\{30\hat{r}\cdot\hat{n}(\hat{n}\times\underline{I}\cdot\hat{r}+\hat{r}\times\underline{I}\cdot\hat{n}) + 15[1-7(\hat{r}\cdot\hat{n})^2]\hat{r}\times\underline{I}\cdot\hat{r} - 6\hat{n}\times\underline{I}\cdot\hat{n}\} \qquad (1)$$

For the sake of giving it a name, body E is assumed to be the Earth so that

$\mu_\oplus$ = gravitational parameter of Earth
R = distance from E^* to B^*
$\hat{r}$ = unit position vector from E^* to B^*
$\underline{I}$ = inertia dyadic of B relative to B^*
J_2 = zonal harmonic coefficient representing Earth's oblateness
$R_\oplus$ = mean equatorial radius of Earth
$\hat{n}$ = unit vector in the direction of Earth's geographic North pole

Derivation

The derivation of Eq. (1) that follows makes extensive use of Ref. 1. Equations from Ref. 1 are denoted by three numbers separated by two decimal points. By way of example, an equation numbered (2.18.1) indicates the first equation appearing in Sec. 18 of Chap. 2 in Ref. 1.

Glandorf[2] seeks a method for constructing gravitational moment expressions from *any* gravitational potential function. Equation (2.18.1) is such a method:

$$M = -\underline{I}\overset{\times}{\cdot}\nabla_R\nabla_R V(R) \qquad (2)$$

where $V(R)$ is a gravitational potential function for a particle of unit mass expressed in terms of a position vector R, and ∇_R denotes differentiation with respect to R. The cross-dot product, $\overset{\times}{\cdot}$, is defined in Ref. 1 and will be discussed in more detail below. Our first task in evaluating Eq. (2) is to obtain an expression for the gravitational potential.

Gravitational Potential

Equation (2.13.14) is an expression for an axisymmetric Earth's gravitational potential for a particle of mass m. The expression contains an infinite series of zonal harmonic coefficients. Since we wish to consider only the zonal harmonic representing Earth's oblateness we can truncate the series and write, for a particle P of unit mass,

$$V_\oplus = \frac{\mu_\oplus}{p}\left[1-\left(\frac{R_\oplus}{p}\right)^2 J_2 P_2(S_\lambda) - \cdots\right] \qquad (3)$$

where

$\mu_\oplus$ = gravitational parameter of Earth
$R_\oplus$ = Earth's mean equatorial radius
p = distance from E^*, the Earth's mass center, to particle P
J_2 = zonal harmonic coefficient representing Earth's oblateness
P_2 = Legendre polynomial of order 2
λ = geographic latitude of P
S_λ = shorthand notation for $\sin\lambda$

The Legendre polynomial, $P_2(S_\lambda)$, is obtained from Eq. (2.13.4):

$$P_2(S_\lambda) = ½[3S_\lambda^2 - 1] = ½\{[3(p\cdot\hat{n})^2/p^2] - 1\} \qquad (4)$$

where

p = position vector from E^* to P
$\hat{n}$ = unit vector in the direction of Earth's geographic North pole

Thus, by substituting Eq. (4) into Eq. (3), we get

$$V_\oplus = \mu_\oplus\left\{\frac{1}{p} - \frac{1}{2}J_2R_\oplus^2\left[\frac{3(p\cdot\hat{n})^2}{p^5} - \frac{1}{p^3}\right]\right\} \qquad (5)$$

Differentiating the Gravitational Potential

Now, Eq. (2) requires expressing $V_\oplus$ in terms of R and differentiating with respect to R. However, p. 150 of Ref. 1 mentions that "differentiating $V(p)$ with respect to p and then setting p equal to R is precisely the same . . .". We can rewrite Eq. (2) as

$$M = -\underline{I}\overset{\times}{\cdot}\nabla_R\nabla_R V(R) = -\underline{I}\overset{\times}{\cdot}[\nabla_p\nabla_p V(p)]_{p\to R} \qquad (6)$$

Equations (2.10.5) and (2.10.6) illustrate how to differentiate terms in which the scalar p appears. Taken together they produce the following result:

$$\nabla_p\frac{1}{p^x} = -x\frac{p}{p^{(x+2)}} \qquad (7)$$

The first differentiation of the gravitational potential with respect to p yields a vector.

$$\nabla_p V_\oplus = \mu_\oplus\left\{-\frac{p}{p^3} - \frac{1}{2}J_2R_\oplus^2\left[\frac{6(p\cdot\hat{n})}{p^5}\nabla_p(p\cdot\hat{n}) - \frac{15(p\cdot\hat{n})^2 p}{p^7} + \frac{3p}{p^5}\right]\right\} \qquad (8)$$

But $\hat{n}$ is independent of p and we can use Eq. (2.9.7) to write

$$\nabla_p(p\cdot\hat{n}) = \underline{U}\cdot\hat{n} = \hat{n} \qquad (9)$$

where $\underline{U}$ is the unit dyadic. Substituting Eq. (9) into Eq. (8) yields the gravitational force acting on P.

$$\nabla_p V_\oplus = \mu_\oplus\left\{-\frac{p}{p^3} - \frac{1}{2}J_2R_\oplus^2\left[\frac{6(p\cdot\hat{n})\hat{n}}{p^5} - \frac{15(p\cdot\hat{n})^2 p}{p^7} + \frac{3p}{p^5}\right]\right\} \qquad (10)$$

The second differentiation of the gravitational potential with respect to p yields a dyadic.

$$\nabla_p \nabla_p V_\oplus = \mu_\oplus \left\{ \frac{3pp}{p^5} - \frac{\underline{U}}{p^3} - \frac{1}{2} J_2 R_\oplus^2 \left[-\frac{30(p \cdot \hat{n})\hat{n}p}{p^7} + \frac{6\hat{n}\hat{n}}{p^5} - \frac{30(p \cdot \hat{n})p\hat{n}}{p^7} - 15(p \cdot \hat{n})^2 \left(-\frac{7pp}{p^9} + \frac{\underline{U}}{p^7} \right) - \frac{15pp}{p^7} + \frac{3\underline{U}}{p^5} \right] \right\} \quad (11)$$

Cross-Dot Identities

At this point we can form the gravitational moment with Eq. (6) by using the dyadic in Eq. (11), after developing identities involving the *symmetric* central inertia dyadic, $\underline{I}$, and the cross-dot product. The cross-dot product, $\overset{\times}{\cdot}$, is defined on p. 156 of Ref. 1. For two dyads, u_1u_2 and v_1v_2,

$$(u_1u_2) \overset{\times}{\cdot} (v_1v_2) \equiv (u_1 \times v_1)(u_2 \cdot v_2) \quad (12)$$

The cross-dot product obeys the distributive law when applied to dyadics.

$$(s_1s_2 + t_1t_2 + \cdots) \overset{\times}{\cdot} (u_1u_2 + v_1v_2 + \cdots) = s_1s_2 \overset{\times}{\cdot} u_1u_2 + s_1s_2 \overset{\times}{\cdot} v_1v_2 + \cdots + t_1t_2 \overset{\times}{\cdot} u_1u_2 + t_1t_2 \overset{\times}{\cdot} v_1v_2 + \cdots + \cdots \quad (13)$$

$\underline{I}$ can be written in terms of inertia vectors, I_j ($j = 1,2,3$).

$$\underline{I} = I_1\hat{b}_1 + I_2\hat{b}_2 + I_3\hat{b}_3 \quad (14)$$

where $\hat{b}_j$ ($j = 1,2,3$) are any mutually orthogonal dexterous unit vectors fixed in B. Inertia vectors can in turn be written in terms of inertia scalars.

$$I_j \equiv I_{j1}\hat{b}_1 + I_{j2}\hat{b}_2 + I_{j3}\hat{b}_3 \qquad (j = 1,2,3) \quad (15)$$

We are now in a position to perform the cross-dot product with $\underline{I}$ and $\underline{U}$. Using Eqs. (12–15),

$$\begin{aligned} \underline{I} \overset{\times}{\cdot} \underline{U} &= (I_1\hat{b}_1 + I_2\hat{b}_2 + I_3\hat{b}_3) \overset{\times}{\cdot} (\hat{b}_1\hat{b}_1 + \hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3) \\ &= I_1 \times \hat{b}_1 + I_2 \times \hat{b}_2 + I_3 \times \hat{b}_3 \\ &= I_{12}\hat{b}_2 \times \hat{b}_1 + I_{13}\hat{b}_3 \times \hat{b}_1 + I_{21}\hat{b}_1 \times \hat{b}_2 + I_{23}\hat{b}_3 \times \hat{b}_2 \\ &\quad + I_{31}\hat{b}_1 \times \hat{b}_3 + I_{32}\hat{b}_2 \times \hat{b}_3 \end{aligned} \quad (16)$$

However, inertia scalars have the property of symmetry

$$I_{jk} = I_{kj} \qquad (j,k = 1,2,3) \quad (17)$$

and, by the definition of a vector cross product,

$$u \times v = -v \times u \quad (18)$$

By substituting Eqs. (17) and (18) into Eqs. (16), it can be seen that

$$\underline{I} \overset{\times}{\cdot} \underline{U} = \mathbf{0} \quad (19)$$

Another useful identity can be obtained with the help of Eq. (14). For any dyad u_1v_1 we can write

$$\begin{aligned} \underline{I} \overset{\times}{\cdot} u_1v_1 &= (I_1\hat{b}_1 + I_2\hat{b}_2 + I_3\hat{b}_3) \overset{\times}{\cdot} u_1v_1 \\ &= (I_1 \times u_1)(\hat{b}_1 \cdot v_1) + (I_2 \times u_1)(\hat{b}_2 \cdot v_1) + (I_3 \times u_1)(\hat{b}_3 \cdot v_1) \\ &= -u_1 \times (I_1\hat{b}_1 \cdot v_1 + I_2\hat{b}_2 \cdot v_1 + I_3\hat{b}_3 \cdot v_1) \\ &= -u_1 \times \underline{I} \cdot v_1 \end{aligned} \quad (20)$$

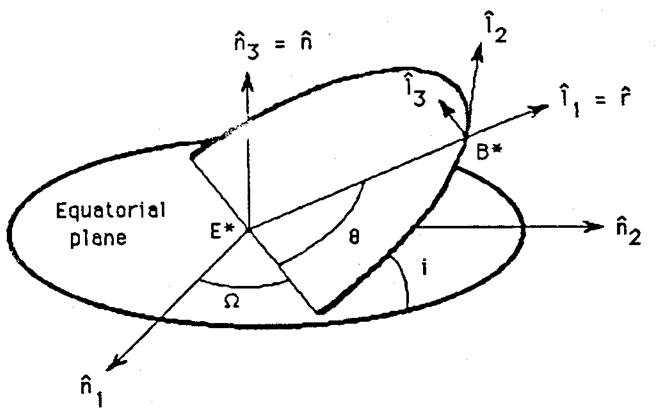

Fig. 2 Orbit of B^* about E.

Gravitational Moment

Now we can replace p with R in Eq. (11), substitute the result into Eq. (6), and make use of identities in Eqs. (19) and (20) to obtain

$$\begin{aligned} M &= \frac{3\mu_\oplus}{R^3}\hat{r} \times \underline{I} \cdot \hat{r} + \mathbf{0} + \frac{\mu_\oplus J_2 R_\oplus^2}{2R^5} [30(\hat{r} \cdot \hat{n})\hat{n} \times \underline{I} \cdot \hat{r} - 6\hat{n} \\ &\quad \times \underline{I} \cdot \hat{n} + 30(\hat{r} \cdot \hat{n})\hat{r} \times \underline{I} \cdot \hat{n} - 15(\hat{r} \cdot \hat{n})^2(7\hat{r} \times \underline{I} \cdot \hat{r} + \mathbf{0}) \\ &\quad + 15\hat{r} \times \underline{I} \cdot \hat{r} + \mathbf{0}] \\ &= \frac{3\mu_\oplus}{R^3}\hat{r} \times \underline{I} \cdot \hat{r} + \frac{\mu_\oplus J_2 R_\oplus^2}{2R^5} \{30\hat{r} \cdot \hat{n}(\hat{n} \times \underline{I} \cdot \hat{r} + \hat{r} \times \underline{I} \cdot \hat{n}) \\ &\quad + 15[1 - 7(\hat{r} \cdot \hat{n})^2]\hat{r} \times \underline{I} \cdot \hat{r} - 6\hat{n} \times \underline{I} \cdot \hat{n}\} \end{aligned} \quad (21)$$

where $\hat{r}$ is the unit position vector from E^* to B^*.

The derivation of Eq. (1) is now complete. Any valid gravitational potential function for a particle of unit mass can be used with Eq. (2) to produce an expression for gravitational moment about a body's mass center. For the case of an oblate spheroid, Eq. (2) has been shown to produce a simple result that does not restrict one to expressing gravitational moment in any particular unit vector basis. Notice that for $J_2 = 0$, Eq. (1) reduces to Eq. (2.6.3), which is the expression for the gravitational moment exerted on B by a sphere (or particle) with the mass of the Earth.

Example

Employing Eq. (1) to express M in a particular basis can quickly become a chore if done algebraically. However, implementing the equation on a computer as part of a numerical simulation is fairly easy. In order to demonstrate the use of Eq. (1) and produce numerical results here, let us consider a very simple example.

Assume that B^*, the mass center of B, is in a circular orbit which is inclined relative to Earth's equatorial plane. Figure 2 shows a reference frame L in which unit vectors $\hat{l}_1$, $\hat{l}_2$, and $\hat{l}_3$ are fixed. Unit vector $\hat{l}_1$ is identical to $\hat{r}$. Unit vector $\hat{l}_2$ is orthogonal to $\hat{l}_1$ and lies in the plane of the orbit of B^*, while $\hat{l}_3$ is perpendicular to the orbit plane. Unit vectors $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ are fixed in frame N such that $\hat{n}_1$ lies in the Earth's equatorial plane in the direction of the vernal equinox, $\hat{n}_2$ also lies in the equatorial plane, and $\hat{n}_3$ is identical to $\hat{n}$, which is perpendicular to the equatorial plane.

Frame L is brought into a general orientation in N by a body-two, 3-1-3 rotation sequence (Ref. 1, p. 424). Unit vectors $\hat{l}_1$, $\hat{l}_2$, and $\hat{l}_3$ are initially aligned with $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$, respectively, and the preceding rotation sequence requires simple rotations of amounts Ω (longitude of ascending node), i (inclination of the orbit plane), and θ (argument of latitude) about $\hat{l}_3$, $\hat{l}_1$, and $\hat{l}_3$, respectively.

The direction cosine matrix on p. 424 of Ref. 1 allows us to write

$$\hat{n} = \hat{n}_3 = S_i S_\theta \hat{l}_1 + S_i C_\theta \hat{l}_2 + C_i \hat{l}_3 \tag{22}$$

where S_i is shorthand for sin i, C_i is short for cos i, etc. Since $\hat{r} = \hat{l}_1$, we also have

$$\hat{r} \cdot \hat{n} = S_i S_\theta \tag{23}$$

Unit vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ are fixed in B, are assumed to be parallel to central principal axes of inertia of B, and are further assumed to be parallel to $\hat{l}_1$, $\hat{l}_2$, and $\hat{l}_3$. Thus, we can express $\underline{I}$ as

$$\begin{aligned} \underline{I} &= I_{11}\hat{b}_1\hat{b}_1 + I_{22}\hat{b}_2\hat{b}_2 + I_{33}\hat{b}_3\hat{b}_3 \\ &= I_{11}\hat{l}_1\hat{l}_1 + I_{22}\hat{l}_2\hat{l}_2 + I_{33}\hat{l}_3\hat{l}_3 \end{aligned} \tag{24}$$

From Eqs. (22) and (24) we can see that

$$\underline{I} \cdot \hat{n} = I_{11} S_i S_\theta \hat{l}_1 + I_{22} S_i C_\theta \hat{l}_2 + I_{33} C_i \hat{l}_3 \tag{25}$$

and

$$\begin{aligned} \hat{n} \times \underline{I} \cdot \hat{n} = {} & (I_{33} - I_{22}) S_i C_i C_\theta \hat{l}_1 + (I_{11} - I_{33}) S_i C_i S_\theta \hat{l}_2 \\ & + (I_{22} - I_{11}) S_i^2 S_\theta C_\theta \hat{l}_3 \end{aligned} \tag{26}$$

similarly

$$\hat{r} \times \underline{I} \cdot \hat{n} = 0\hat{l}_1 - I_{33} C_i \hat{l}_2 + I_{22} S_i C_\theta \hat{l}_3 \tag{27}$$

$$\hat{r} \times \underline{I} \cdot \hat{r} = \mathbf{0} \tag{28}$$

$$\hat{n} \times \underline{I} \cdot \hat{r} = 0\hat{l}_1 + I_{11}(C_i \hat{l}_2 - S_i C_\theta \hat{l}_3) \tag{29}$$

After substitution of Eqs. (23) and (26–29) into Eq. (1), we find that the body-basis measure numbers of $\boldsymbol{M}$ are

$$\boldsymbol{M} \cdot \hat{b}_1 = \boldsymbol{M} \cdot \hat{l}_1 = \frac{3\mu_\oplus J_2 R_\oplus^2}{R^5}(I_{22} - I_{33}) S_i C_i C_\theta \tag{30}$$

$$\boldsymbol{M} \cdot \hat{b}_2 = \boldsymbol{M} \cdot \hat{l}_2 = \frac{12\mu_\oplus J_2 R_\oplus^2}{R^5}(I_{11} - I_{33}) S_i C_i S_\theta \tag{31}$$

$$\boldsymbol{M} \cdot \hat{b}_3 = \boldsymbol{M} \cdot \hat{l}_3 = \frac{12\mu_\oplus J_2 R_\oplus^2}{R^5}(I_{22} - I_{11}) S_i^2 S_\theta C_\theta \tag{32}$$

In order to obtain an appreciation for numerical values of these measure numbers, we will consider central principal moments of inertia that are commensurate with a vehicle the size of a space station, and a radius of a conceivable circular space station orbit. The following values of astronomical constants for Earth, central principal moments of inertia, and orbital parameters are used:

$\mu_\oplus = 3.986 \times 10^5$ km^3/s^2
$J_2 = 1.08 \times 10^{-3}$
$R_\oplus = 6378$ km
$I_{11} = 1.355 \times 10^8$ kg-m^2
$I_{22} = I_{11}$
$I_{33} = \frac{1}{10} I_{11}$
$i = 28.5$ deg
$R = 6700$ km

We find that the measure numbers of the gravitational moment exerted about the mass center of our vehicle would vary with argument of latitude in the following manner:

$\boldsymbol{M} \cdot \hat{b}_1 = (0.1991 \text{ N-m}) C_\theta$
$\boldsymbol{M} \cdot \hat{b}_2 = (0.7964 \text{ N-m}) S_\theta$
$\boldsymbol{M} \cdot \hat{b}_3 = 0$

These numerical values become significant when one considers that moments of aerodynamic forces about the space station mass center are predicted to be in the neighborhood of 2 N-m, and the first term in Eq. (1) can produce values near 3 N-m for orientations of B in L other than the one considered in this example.

Conclusions

A method for obtaining vector-dyadic expressions for the gravitational moment about a body's mass center has been demonstrated. The demonstration has been accomplished by deriving an expression for the gravitational moment exerted by an oblate spheroid. The contribution of Earth oblateness to the gravitational moment exerted on a body has been evaluated numerically in a greatly simplified example. This contribution is significant in comparison with other external moments, such as the one produced by aerodynamic forces.

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Tracking Accuracy for LEOSAT-GEOSAT Laser Links

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Introduction

THE importance of laser-based intersatellite communication links (LASERCOM) has emerged recently. Keeping in mind its advantages, several workers[1–3] have reported on the antenna diameters and the pointing, acquisition, and tracking (PAT) accuracy requirements of laser intersatellite links. Particularly with regard to laser-diode-based intersatellite links, Boutemy et al.[2] have assumed a pointing accuracy of 1 μrad for the low-Earth-orbit to geostationary-orbit link. Popescu et al.[3] have postulated a beam divergence of 3 μrad.

As the actual values of the beam divergence and the tracking accuracy requirements for various antenna diameters and signal-to-noise ratios (S/N) have to be known for design of the LASERCOM, the values are calculated using the Gaussian beam characteristics of the laser.

Methodology

Since the laser beam has a Gaussian intensity profile, the following relations[4] hold:

$$w^2(z) = w_o^2[1 + (\lambda z / \pi w_o^2)^2] \tag{1}$$

$$\theta = \lambda / \pi w_o \tag{2}$$

Received March 25, 1988; revision received May 6, 1988.

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